Introduction to the Banach-Saks Property

hOREP

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Contents

1 Introduction

The Banach-Saks property relates boundedness to convergence under the norm of the arithmetic mean, for a Banach space.

Definition 1.1. *A Banach space X is said to have the Banach-Saks property if every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ contains a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ such that [1]*

$$s_k = \frac{1}{k}\sum_{j=1}^{k} x_{n_j}, \tag{1}$$

is norm convergent.

We will show that every Hilbert space has the Banach-Saks property. We will need several definitions and theorems to reach this. First, we define weak convergence for a Banach space and state Banach's Theorem. We then prove that Hilbert spaces have the Banach-Saks property, and then discuss other spaces that also have this property.

2 Weak Convergence

Let X be a Banach space. Typically convergence, that is norm convergence where $\|x - x_n\| \to 0$ as $n \to \infty$ for some sequence $(x_n) \subset X$ is too strong of a requirement. When working in finite dimensional spaces, the Bolzano-Weierstrass Theorem guarantees that a bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. When moving to infinite dimensions this is no longer the case. If we want a similar result we instead move to weak convergence. Weak convergence depends upon the use of the dual space, denoted $X^* = \mathcal{B}(X, \mathbb{R})$, the set of bounded linear functionals from the set X to the reals. The second dual $X^{**} = (X^*)*$ is the set of bounded linear functions from X^* to $\mathbb{R}$.

Definition 2.1. *A sequence $(x_n) \subset X$ in a normed vector space $(X, \|\cdot\|)$ is said to* converge weakly *to $x_0 \in X$ if as $n \to \infty$ we have [6, p.257],[10, p.90]*

$$x^*(x_n) \to x^*(x_0), \quad \forall x^* \in X^*. \tag{2}$$

This is denoted $x_n \rightharpoonup x_0$ as $n \to \infty$ and x_0 is called the weak limit *of (x_n).*

Weak convergence is called "weak" because norm convergence (strong convergence) implies weak convergence, but not the other way around. Let $(x_n) \subset X$ be such that $x_n \to x_0$ as $n \to \infty$ for $x_0 \in X$ and let $x^* \in X^*$. We have by linearity

$$|x^*(x_n) - x^*(x_0)| = |x^*(x_n - x_0)|. \tag{3}$$

Noting that x^* is bounded, we can choose $M > 0$ such that $|x^*(x_n - x_0)| \leq M \|x_n - x_0\|$ so

$$|x^*(x_n) - x^*(x_0)| \leq M \|x_n - x_0\| \to 0, \quad n \to \infty. \tag{4}$$

It follows that $x_n \rightharpoonup x_0$.

We would like a replacement for the Bolzano-Weierstrass theorem for infinite dimensions. To get to Banach's Theorem, we introduce reflexivity.

Definition 2.2. *The* canonical mapping of X into X^{**} *is defined by [14, p.276]*

$$\hat{x}(x^*) = x^*(x), \tag{5}$$

for $x \in X, x^* \in X^*$. The space $\hat{X}$ are the elements generated by the canonical mapping, i.e. $\hat{X} = \hat{x}(X)$ [6, p.107-108].

The canonical mapping gives us a way to generate elements of X^{**} from elements of X, so it is immediate that $\hat{X} \subset X^{**}$. The question is whether or not this mapping gives us all elements of X^{**}, and if it does the space X is said to be reflexive.

Definition 2.3 (Reflexivity)**.** *If* $\hat{X} \equiv X^{**}$ *then* X *is called reflexive [6, p.241] [9].*

We can now state Banach's Theorem, which can be proven via the Banach-Alaoglu Theorem [15].

Theorem 2.1 (Banach)**.** *Let* $(X, \|\cdot\|)$ *be a reflexive Banach space. Then any bounded sequence* $(x_n) \subset X$ *has a weakly convergent subsequence* $(x_{n_k}) \subset (x_n)$ *such that* $x_{n_k} \rightharpoonup x_0 \in X$ *as* $k \to \infty$*.*

This replaces the Bolzano-Weierstrass Theorem in infinite dimensions. For example, the sequence $(x_n) \subset$

3 Hilbert Spaces

Due to the Riesz Representation Theorem [18] weak convergence takes a particularly nice form in a Hilbert space H, and we also get reflexivity of H.

Theorem 3.1 (Riesz-Fréchet)**.** *For every bounded linear functional* $f \in H^*$*, there exists a unique* $z \in H$ *such that [6, p.188],[18, p.62],[14, p.313]*

$$f(x) = \langle x, z \rangle \tag{6}$$

for every $x \in H$*, satisfying*

$$\|z\| = \|f\|. \tag{7}$$

The definition of weak convergence within a Hilbert space is then as follows.

Definition 3.1. *A sequence* $(x_n) \subset H$ *in a Hilbert space* $(H, \langle \cdot, \cdot \rangle)$ *converges weakly to* $x \in H$ *if and only if [17],[7, p.127],[3, p.98]*

$$\langle x_n, z \rangle \to \langle x, z \rangle \tag{8}$$

as $n \to \infty$ *for every* $z \in H$*.*

A typical example of a weakly convergent sequence is any orthonormal sequence. A sequence (u_n) is orthonormal if $\langle u_n, u_m \rangle = 0$ if $n \neq m$, and 1 if $n = m$. All orthonormal sequences tend weakly to zero by Bessel's inequality [18, p.34], as for every $z \in H$ we have

$$\sum_{n=1}^{\infty} |\langle u_n, z \rangle|^2 \leq \|z\|^2 \tag{9}$$

from which we deduce the sum is convergent, hence $|\langle u_n, z \rangle|^2 \to 0$ as $n \to \infty$, that is $\langle u_n, z \rangle \to 0 = \langle 0, z \rangle$. It is also obvious that (u_n) does not converge "strongly", in terms of the norm, as for $n \neq m$ we have (by Pythagoras) $\|u_n - u_m\|^2 = \|u_n\|^2 + \|u_m\|^2 = 2$, so $\|u_n - u_m\| = \sqrt{2}$. It follows that (u_n) isn't Cauchy, so cannot have any convergent subsequences. This shows that weak convergence does not imply strong convergence.

3.1 Banach-Saks Property

Every Hilbert space has the Banach-Saks property [11]. We show this, following the proof given in [14, p.314].

Theorem 3.2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert Space. Then for any bounded sequence in H there exists a subsequence such that the Cesàro means converge in H.*

Proof. Let $(x_j)_{j\in\mathbb{N}} \subset H$ be bounded. Then, noting H is reflexive, by Theorem 2.1 there exists a weakly convergent subsequence (x_{j_n}) such that $x_{j_n} \to x_0 \in H$ as $n \to \infty$. Now define $u_n = x_{j_n} - x_0$ for each n, which clearly converges weakly to zero.

Since (u_n) converges weakly, it is bounded, and there exists some $M > 0$ such that

$$\|u_n\|^2 \leq M \tag{10}$$

for every $n \in \mathbb{N}$. We now want to choose a subsequence (u_{n_k}) such that for every k,

$$\|u_{n_1} + \cdots + u_{n_k}\|^2 \leq (2 + M)k \tag{11}$$

so that dividing through by k^2 yields

$$\left\| \frac{u_{n_1} + \cdots + u_{n_k}}{k} \right\|^2 \leq \frac{2 + M}{k}. \tag{12}$$

This will be done using induction.

Choose $n_1 = 1$. Since $u_n \rightharpoonup 0$, and $u_{n_1} \in H$, we can choose $n_2 \in \mathbb{N}$ such that $|\langle u_{n_1}, u_{n_2}\rangle| < 1$. Now since $u_n \rightharpoonup 0$ and $u_{n_1}, u_{n_2} \in H$, we can choose $n_3 > n_2$ such that both $|\langle u_{n_1}, u_{n_3}\rangle|, |\langle u_{n_2}, u_{n_3}\rangle| < 1$. Then

$$\begin{aligned}
\|u_{n_1} + u_{n_2} + u_{n_3}\|^2 &= \langle u_{n_1} + u_{n_2} + u_{n_3}, u_{n_1} + u_{n_2} + u_{n_3}\rangle && (13)\\
&= \langle u_{n_1} + u_{n_2} + u_{n_3}, u_{n_1}\rangle + \langle u_{n_1} + u_{n_2} + u_{n_3}, u_{n_2}\rangle + \langle u_{n_1} + u_{n_2} + u_{n_3}, u_{n_3}\rangle && (14)\\
&= \langle u_{n_1}, u_{n_1}\rangle + \langle u_{n_2}, u_{n_1}\rangle + \langle u_{n_3}, u_{n_1}\rangle \\
&+ \langle u_{n_1}, u_{n_2}\rangle + \langle u_{n_2}, u_{n_2}\rangle + \langle u_{n_3}, u_{n_2}\rangle \\
&+ \langle u_{n_1}, u_{n_3}\rangle + \langle u_{n_2}, u_{n_3}\rangle + \langle u_{n_3}, u_{n_3}\rangle && (15)\\
&\leq M + 1 + 1 && (16)\\
&+ 1 + M + 1 \\
&+ 1 + 1 + M. && (17)
\end{aligned}$$

We have three diagonal terms, and using (10) we know these are each less than M. For the off diagonal terms, we have chosen n_1, n_2, n_3 so that these are all less than 1 in magnitude, and there are six of them. This gives the upper bound

$$\|u_{n_1} + u_{n_2} + u_{n_3}\|^2 \le 6 + 3M = (2+M)3 \tag{18}$$

For the inductive step, assume that we have chosen natural numbers $n_1 < n_2 < \cdots < n_k$ such that

$$\|u_{n_1} + \cdots + u_{n_k}\|^2 \le (2+M)j \tag{19}$$

for $j = 1, \ldots, k$. Since $u_n \rightharpoonup 0$ and $u_{n_1} + \cdots + u_{n_k} \in H$ we can choose $n_{k+1} > n_k$ such that

$$|\langle u_{n_1} + \cdots + u_{n_k}, u_{n_{k+1}} \rangle| \le 1. \tag{20}$$

Then

$$\left\|u_{n_1} + \cdots + u_{n_{k+1}}\right\|^2 = \|u_{n_1} + \cdots + u_{n_k}\|^2 + 2\mathrm{Re}\left(\langle u_{n_1} + \cdots + u_{n_k}, u_{n_{k+1}}\rangle\right) + \left\|u_{n_{k+1}}\right\|^2 \tag{21}$$

$$\le (2+M)k + 2 + M \tag{22}$$

$$= (2+M)(k+1). \tag{23}$$

We now have a subsequence that satisfies

$$\left\|\frac{u_{n_1} + \cdots + u_{n_k}}{k}\right\|^2 \le \frac{2+M}{k} \tag{24}$$

for each k. We can now recover our original sequence, as $u_{n_k} = x_{j_{n_k}} - x_0$, and we have

$$\left\|\frac{x_{j_{n_1}} + \cdots + x_{j_{n_k}}}{k} - x_0\right\|^2 = \left\|\frac{x_{j_{n_1}} - x_0 + \cdots + x_{j_{n_k}} - x_0}{k}\right\|^2 \le \frac{2+M}{k}. \tag{25}$$

Letting $k \to \infty$ yields the result. □

3.2 Examples

We will take an orthonormal sequence as an example. Let $(u_n) \subset H$ be an orthonormal sequence, and hence bounded. We can simply take $n_k = k$ as the subsequence in this case. We have

$$\|s_k\|^2 = \left\|\frac{1}{k}\sum_{j=1}^{k} u_j\right\|^2 = \frac{1}{k^2}\left\|\sum_{j=1}^{k} u_j\right\|^2 = \frac{1}{k^2}\sum_{j=1}^{k}\|u_j\|^2 = \frac{1}{k^2}\sum_{j=1}^{k} 1 = \frac{1}{k}, \tag{26}$$

where we have used the Pythagorean Theorem. It follows that

$$\frac{1}{k}\sum_{j=1}^{k} u_j \to 0 \tag{27}$$

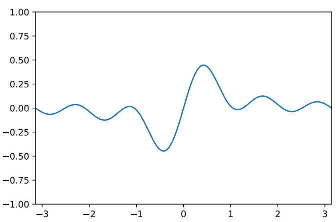
as $k \to \infty$. For a more concrete example, take the Hilbert space $L^2[-\pi, \pi]$ with functions $(f_n) \subset L^2[-\pi, \pi]$ defined by

$$f_n(t) = \frac{1}{\sqrt{\pi}} \sin(nt), \quad t \in [-\pi, \pi]. \tag{28}$$

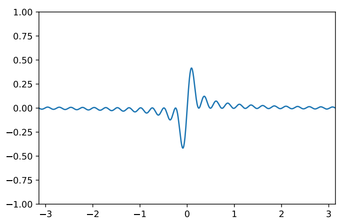
It is easily shown that (f_n) is orthonormal. It then follows that

$$\frac{1}{\sqrt{\pi}k}\sum_{n=1}^{k} \sin(nt) \to 0 \tag{29}$$

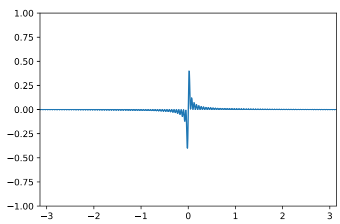
in terms of norm. The Cesàro sum s_k is shown in Figures 1a, 1b, 1c for $k = 5, 25, 125$ respectively.



(a) $k = 5$. The integral is $1/\sqrt{5}$.



(b) $k = 25$. The integral is $1/5$.



(c) $k = 125$. The integral is $1/\sqrt{125}$.

Figure 1: The graph of $s_k(t)$ over $[-\pi, \pi]$ for $k = 5, 25, 125$. The integrals tend to 0 as $n \to \infty$.

4 Uniformly Convex Banach Spaces

James A. Clarkson introduced the idea of a uniformly convex Banach space in his 1936 paper [2]. He states the property geometrically as "the midpoint of a variable chord of the unit sphere of the space cannot approach the surface of the sphere unless the length of the chord goes to zero.” which can be seen in Figure 2. Clearly the midpoint of the chord will only be close the circle boundary when x and y get sufficiently close together.

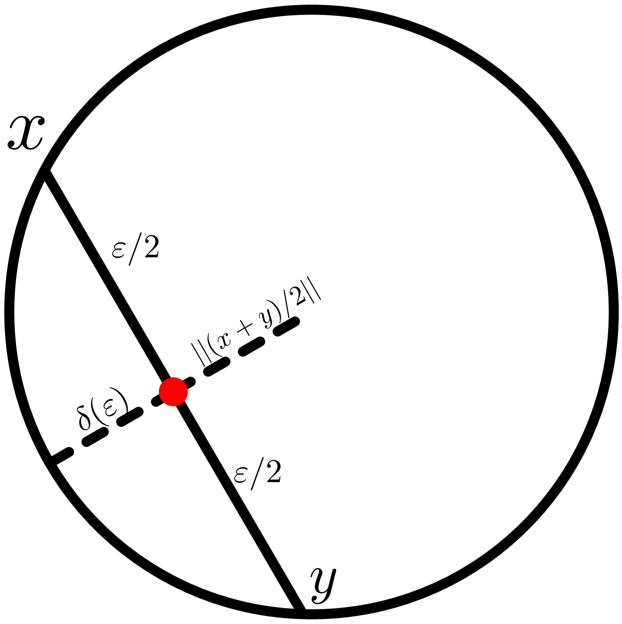


Figure 2: Geometric interpretation of uniformly convex for a $2d$ circle.

The rigorous definition is as follows.

Definition 4.1. *A Banach space X is said to be* uniformly convex *if for each $0 < \varepsilon \leq 2$, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions [2]*

$$\|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \tag{30}$$

imply

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon). \tag{31}$$

It will be useful to note that uniformly convex Banach spaces are reflexive, which follows by the Milman-Pettis theorem [8],[13]. An alternative and shorter proof is given by Ringrose [12].

4.1 Hilbert Spaces

Finite and infinite dimensional Hilbert spaces are uniformly convex, due to the parallelogram law [16]

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 . \tag{32}$$

Setting the required conditions $\|x\| = \|y\| = 1$, dividing by 4 and isolating the $\|x+y\|^2$ term yields

$$\left\|\frac{x+y}{2}\right\|^2 + \frac{1}{4}\|x-y\|^2 = 1. \tag{33}$$

Letting x, y satisfy $\|x-y\| \geq \varepsilon$,

$$\left\|\frac{x+y}{2}\right\|^2 = 1 - \frac{1}{4}\|x-y\|^2 \leq 1 - \frac{1}{4}\varepsilon^2. \tag{34}$$

We have chosen δ to be

$$\delta(\varepsilon) = \frac{1}{4}\varepsilon^2 = \left(\frac{\varepsilon}{2}\right)^2 . \tag{35}$$

The term $\varepsilon/2$ will appear in the next section.

4.2 L^p and l^p Spaces

Now, Clarkson showed that the spaces L^p and l^p are uniformly convex [2]. Hanner expanded on this by providing a simpler set of inequalities to prove this in [4]. The L^p space considered here is $L^p[0,1]$.

Theorem 4.1. *For $p > 2$ the following inequalities hold*

$$(\|x\| + \|y\|)^p + |\|x\| - \|y\||^p \geq \|x+y\|^p + \|x-y\|^p \geq 2\|x\|^p + 2\|y\|^p , \tag{36}$$

holding in the reverse sense for $p \in (1,2)$. The equality sign holds for L^p [for l^p] on the left-hand side if and only if $x = 0$, or $y = 0$, or there exists a number $a > 0$ such that $(x(t) - ay(t))(x(t) + ay(t)) = 0$ for almost every $t \in [0,1]$ [such that $(x_i - ay_i)(x_i + ay_i)$ for every i], and in the right-hand side if and only if $x(t)y(t) = 0$ for almost every $t \in [0,1]$ [$x_i y_i = 0$ for every i].

This is proven in [2]. If $p = 2$ we retrieve equation (32), with equality between all three terms. Now, to show that the mentioned spaces are uniformly convex, we have the following theorem.

Theorem 4.2. *Let x and y be two elements of L^p or l^p such that*

$$\|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon \tag{37}$$

for $0 < \varepsilon < 2$. Then

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon), \tag{38}$$

where $\delta = \delta(\varepsilon)$ is determined by

$$2 = \left(1 - \delta + \frac{\varepsilon}{2}\right)^p + \left|1 - \delta - \frac{\varepsilon}{2}\right|^p, \quad 1 < p < 2 \tag{39}$$

$$\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, \quad p \geq 2. \tag{40}$$

For each ε, x and y can be chosen so that equality holds in (38).

4.3 Banach-Saks Property

In [5] it is shown that a uniformly convex Banach space has the Banach-Saks property. We include this proof here. Kakutani introduces an equivalent formulation of uniform convexity, stated in the following proposition.

Proposition 4.1. *A Banach space X is uniformly convex if and only if for each $\varepsilon > 0$ there corresponds a $\delta'(\varepsilon) > 0$ such that the condition*

$$\|x - y\| \geq \varepsilon \cdot \max\left(\|x\|, \|y\|\right) \tag{41}$$

implies

$$\left\| \frac{x+y}{2} \right\| \leq \left(1 - \delta'(\varepsilon)\right) \cdot \max\left(\|x\|, \|y\|\right). \tag{42}$$

This is a useful reformulation, and can be used to prove the following.

Theorem 4.3. *If a Banach space is uniformly convex, it has the Banach-Saks property [5].*

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