Reciprocal Gamma Integrals

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1 Introduction

Define the Gamma function [1] by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}. \tag{1.1}$$

The Fransén–Robinson constant F is defined as [2]

$$F = \int_0^\infty \frac{1}{\Gamma(x)}\, \mathrm{d}x \approx 2.8077702420285\cdots \tag{1.2}$$

and is noticeably close to Euler's number $e \approx 2.71828\cdots$. This is explained by

$$F = \int_0^\infty \frac{1}{\Gamma(x)}\, \mathrm{d}x \approx \sum_{n=0}^{\infty} \frac{1}{n!} = e, \tag{1.3}$$

and the error can be written explicitly as

$$F - e = \int_0^\infty \frac{e^{-x}}{\pi^2 + (\ln x)^2}\, \mathrm{d}x. \tag{1.4}$$

However, this is not the only extension of the factorial $n!$, as explained by Peter Luschny [3] on his website. We define the Digamma function ψ by

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.5}$$

Then the Hadamard Gamma function H is defined by

$$H(x) = \frac{1}{\Gamma(1-x)} \frac{\mathrm{d}}{\mathrm{d}x} \ln\left(\frac{\Gamma(1/2 - x/2)}{\Gamma(1 - x/2)}\right) \tag{1.6}$$

or equivalently with ψ,

$$H(x) = \frac{\psi(1 - x/2) - \psi(1/2 - x/2)}{2\Gamma(1-x)}. \tag{1.7}$$

and

$$H(x) = \Gamma(x)\left[1 + \frac{\sin(\pi x)}{2\pi}\left\{\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right\}\right]. \tag{1.8}$$

Are there similar constants to F with other gamma functions? Do they satisfy similar properties?

2 Hadamard Reciprocal Integral

We are interested in the value of the integral

$$\hat{H} = \int_0^\infty \frac{1}{H(x)}\,\mathrm{d}x, \tag{2.1}$$

that is,

$$\hat{H} = \int_0^\infty \frac{1}{\Gamma(x)\left[1 + \frac{\sin(\pi x)}{2\pi}\left\{\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right\}\right]}\,\mathrm{d}x. \tag{2.2}$$

Using the mpmath PYTHON library mp.quad function over $(0, \infty)$, the integral has approximate value

$$\hat{H} = \int_0^\infty \frac{1}{H(x)}\,\mathrm{d}x \approx 3.368202929607. \tag{2.3}$$

This value is very close to

$$3 + \frac{1}{e} = 3.3678794411171442\cdots. \tag{2.4}$$

This suggests a similar identity to Equation 1.4. The value of this integral can be similarly approximated using the sum

$$\hat{H} = \int_0^\infty \frac{1}{H(x)}\,\mathrm{d}x \approx \sum_{n=0}^{\infty} \frac{1}{H(n)}. \tag{2.5}$$

Now, $H(n) = n!$ for $n = 1, 2, 3, 4, \cdots$ and

$$H(0) = \frac{\psi(1) - \psi(1/2)}{2\Gamma(1)} = \frac{-\gamma - (-\gamma - \ln(4))}{2} = \frac{\ln(4)}{2} = \ln(2). \tag{2.6}$$

It follows that

$$\sum_{n=0}^{\infty} \frac{1}{H(n)} = \frac{1}{\ln(2)} + \sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{\ln(2)} + e - 1 \approx 3.16097686934800\cdots. \tag{2.7}$$

3 Luschny Reciprocal Integral

Define

$$g(x) = \frac{x}{2}\left[\psi\left(0, \frac{x+1}{2}\right) - \psi\left(0, \frac{x}{2}\right)\right] - 1/2, \tag{3.1}$$

and

$$P(x) = 1 - g(x)\frac{\sin(\pi x)}{\pi x}. \tag{3.2}$$

Then the Luschny Factorial L is defined as [3]

$$L(x) = \Gamma(x+1)P(x). \tag{3.3}$$

We now seek the value of

$$I = \int_0^\infty \frac{1}{L(x)}\,\mathrm{d}x. \tag{3.4}$$

Applying the sum argument again, we have

$$\int_0^\infty \frac{1}{L(x)}\,\mathrm{d}x \approx \sum_{n=0}^\infty \frac{1}{L(n)} \tag{3.5}$$

and we know that $L(n) = n!$ for $n = 1, 2, 3, \cdots$, and $L(0) = 1/2$. We then get

$$\sum_{n=0}^\infty \frac{1}{L(n)} = 2 + e - 1 = e + 1 \approx 3.71, \tag{3.6}$$

however applying the mpmath quadrature we get

$$\int_0^\infty \frac{1}{L(x)}\,\mathrm{d}x \approx 2.586705059786808227\cdots. \tag{3.7}$$

This is a significant deviation! Clearly this simple approach will not work. We can instead apply the Abel-Plana formula [4] to reach a similar approximation. Assuming $1/L(x)$ is "nice enough" we have

$$\begin{aligned}\int_0^\infty \frac{1}{L(x)}\,\mathrm{d}x &= -\frac{1}{2L(0)} + \sum_{n=0}^\infty \frac{1}{L(n)} - i\int_0^\infty \frac{1/L(ix) - 1/L(-ix)}{e^{2\pi x} - 1}\,\mathrm{d}x & (3.8)\\ &= e - i\int_0^\infty \frac{1/L(ix) - 1/L(-ix)}{e^{2\pi x} - 1}\,\mathrm{d}x & (3.9)\\ &\approx e. & (3.10)\end{aligned}$$

This is much closer, and the above relation can be "numerically verified to not be immediately false".

4 Generalised Luschny Reciprocal Integral

Define

$$g(x,\alpha) = \frac{x}{2}\left[\psi\left(0, \frac{x+1}{2}\right) - \psi\left(0, \frac{x}{2}\right)\right] - \alpha/2, \tag{4.1}$$

and

$$P(x,\alpha) = 1 - g(x,\alpha)\frac{\sin(\pi x)}{\pi x}. \tag{4.2}$$

Then the Generalised Luschny Factorial $L(x,\alpha)$ is defined as [3]

$$L(x,\alpha) = \Gamma(x+1)P(x,\alpha). \tag{4.3}$$

Then it easily shown that

$$L(x) = L(x,1), \quad H(x+1) = L(x,2). \tag{4.4}$$

Define

$$\mathcal{I}(\alpha) = \int_0^\infty \frac{1}{L(x,\alpha)}\,\mathrm{d}x. \tag{4.5}$$

Graphing the parameter α against I yields Figure 1.

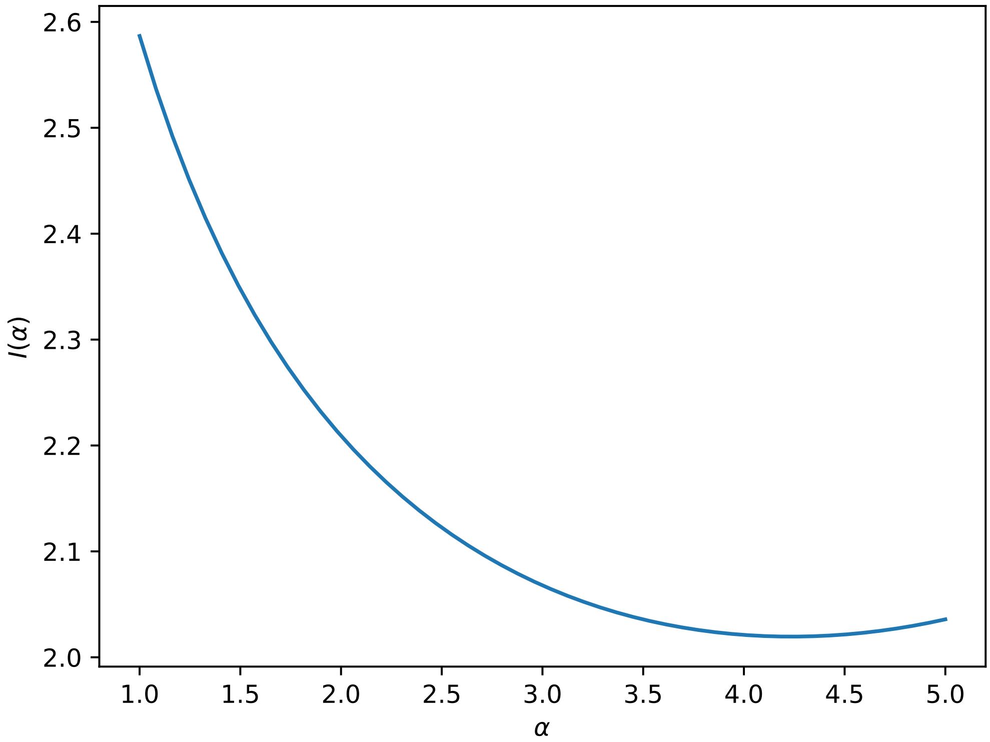

Figure 1. Relation between $\mathcal{I}(\alpha)$ and the input parameter α, over the interval $(1, 5)$. Note the appearance of a minimum between 4 and 4.5.

References

[1] Weisstein, Eric W. "Gamma Function." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/GammaFunction.html

[2] Weisstein, Eric W. "Fransén-Robinson Constant." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Fransen-RobinsonConstant.html

[3] Luschny, Peter. Hadamard's Gamma Function. http://www.luschny.de/math/factorial/hadamard/HadamardsGammaFunctionMJ.html

[4] Anderson, David. "Abel-Plana Formula." From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. https://mathworld.wolfram.com/Abel-PlanaFormula.html